Written re-exam on Lab-on-a-Chip course, Spring Semester 2009
Aug 26th, 2009

Examination time: 4 hours (9am – 1pm)
Allowed means: MYO "Fundamentals of Fluid Mechanics", lecture slides and a calculator (no computer or PDAs, please)
Complete solution should include all equations calculated down to a numerical answer. Numerical answer alone is not counted as a solution.
Some useful constants for your problems are listed in the end of the exam paper

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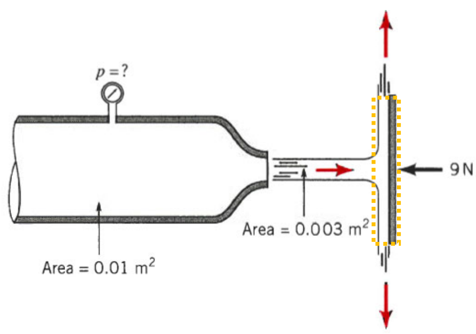
Problem 1. A circular stream of water flowing down from a faucet is observed to taper from a diameter 20mm to 10mm in a distance of 50mm. Determine the flow rate.

Solution:
From Bernoulli equation
$\frac{1}{2}\rho V_1^2 + \rho g h = \frac{1}{2}\rho V_2^2$, where ρ is density of water, h – height difference and V_1 and V_2 are the velocities in upper and lower cross sections correspondingly. We take into account that the pressure in both cross-sections is the same and equal to the atmospheric one.
From the continuity and assuming water incompressible:
$\pi r_1^2 V_1 = \pi r_2^2 V_2$, where A_1 and A_2 are the corresponding cross-sections. As the lower diameter is twice as low as the upper one, we can conclude that $V_1 = V_2/4$
Inserting it back into Bernoulli equations produces:
$V_1 = \sqrt{\frac{2}{15} g h} = 0.26\, m/s$ and the flow rate is $\pi r_1^2 V_1 = \pi r_2^2 V_2 = 8 \cdot 10^{-5} \frac{m^3}{s}$



Problem 2. Air flows into the atmosphere from a nozzle and strikes a vertical plate. A horizontal force of 9N is required to hold the plate in place. Determine the reading on the pressure gauge. Assume the flow to be incompressible and frictionless..

Solution:
Let's choose a control volume as shown in the figure. From the Reynolds theorem, the force is equal to the net flow of the

momentum:
$-V_2 \rho V_2 A_2 = -F$.
Taking the density of air at atmospheric pressure as 1.23 kg/m^3 (table 1.8, MYO book cover) we find the velocity at the outlet: 49.4 m/s.
Now we write Bernoulli and continuity equations:

$$\frac{1}{2}\rho_1 V_1^2 + P_1 = \frac{1}{2}\rho_2 V_2^2$$
$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

As the air is compressible, the density is different at the sections 1 and 2. The easiest way to solve the problem is to assume that the effect is negligible and treat the air as incompressible. This will be justified as the pressure difference you will find in the answer is small. Than:

$$P = \frac{1}{2}\rho V_2^2 \left[1 - \left(\frac{A_2}{A_1}\right)^2\right] = 1.4\,kPa$$

Problem 3. The two-dimensional velocity field for an incompressible Newtonian fluid is described as:
$\vec{V} = (12xy^2 - 6x^3)\vec{i} + \left(18x^2 y - 4y^3\right)\vec{j}$

a) write the relationship for acceleration and determine it at the point with coordinates (0.5m, 1.0m)
b) Determine the stresses $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$ at the same point if the pressure at this point is 6kPa and the fluid is glycerin at 20ºC.

Solution:

a) Acceleration:

$\vec{a} = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$, so the x- and y- components are

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}; a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y};$$

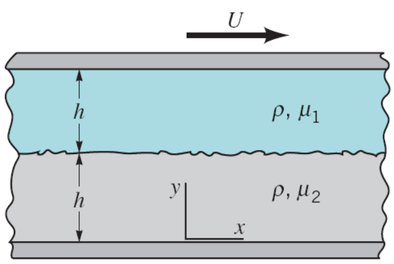
After performing differentiation and inserting the coordinates you will find:
$a_x = 45.4\,\text{m/s}^2; a_y = 90.8\,\text{m/s}^2$

b) Stresses:

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} = -6\cdot 10^3 + 2\cdot 1.50\left(12\cdot 1^2 - 18\cdot 0.5^2\right) = -5.98kPa$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y} = -6.02kPa$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 45\,Pa$$



Problem 4. Two immiscible incompressible viscous fluids having the same densities but different viscosities are contained between two infinite, horizontal plates. The bottom plate is fixed and the upper plates moves with a constant velocity U. Determine the velocity at the interface using Navier-Stokes equation. Assume laminar flow and no pressure gradient in x direction.

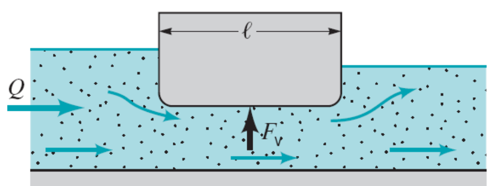
<u>Solution:</u>
For each of the layers the Navier-Stokes equation will be reduced to:
$\frac{d^2u}{dy^2} = 0$, leading to a linear velocity distribution in each layer (as expected for Couette flow): $u = Ay + B$, where A and B should be found from the boundary conditions for each layer: $u_2(0)=0$; $u_2(h)=u_1(h)$; $u_1(2h)=0$.
In addition the stress should be continuous at the layer interface:

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\frac{\partial u}{\partial y},$$ yielding

$$\mu_1 A_1 = \mu_2 A_2$$

Solving the linear equation, we find:

$$U(h) = \frac{U}{1 + \frac{\mu_2}{\mu_1}}$$



Problem 5. Water flowing under the obstacle puts a vertical force $\mathbf{F}_v$ on the obstacle. This force is assumed to be a function

of flow rate **Q**, the density of water **ρ**, the acceleration of gravity **g**, and a length ***l***, that characterizes the size of obstacle.
a) Find equation for the force using dimensional analysis and the above mentioned parameters.
b) A 1/20 scale model is built. If the prototype flow rate is 28 m^3/s, what flow rate is required in the model to be similar.

<u>Solution:</u>
We use dimensional analysis to find the dependence:
$F = f(Q, \rho, g, l)$
There are 5 variables and all 3 dimensions are used, so from the Pi-theorem, two Pi-terms are required. From the dimensional analysis we find:

$$\frac{F_V}{\rho g l^3} = \Phi\left(\frac{Q^2}{g l^5}\right)$$

The models are similar when the Pi terms are equal. Here it is convenient to equate Π_2 terms as they only contain parameters of interest ***Q*** and ***l*** and a constant ***g***.

$$\frac{Q_m^{\ 2}}{l_m^{\ 5}} = \frac{Q^2}{l^5}$$

If ***l*** is changed by a factor of 1/20, the ***Q*** should be 0.016 m^3/s.

Problem 6. An experimentalist wants to achieve a water flow of 20 µl/min. He has a pressure regulator capable of applying 1 atm of gauge pressure and tubing of 0.125, 0.25 and 0.5mm diameter. What length of each type of tubing would be required to achieve the required flow rate?

<u>Solution:</u>
For a laminar flow in pipes:

$$Q = \frac{\pi D^4 \Delta p}{128 \mu l} \quad \text{or} \quad l = \frac{\pi D^4 \Delta p}{128 \mu Q}$$

Inserting the numbers we find:

$$D = 0.125mm \quad l = 1.8m$$

$$D = 0.25mm \quad l = 29m$$

$$D = 0.5mm \quad l = 465\,m$$

This shows how strongly flow rate (or pressure) depends on the tube diameter.
A common mistake here is in the unit conversion. Please remember:

$$1\mu l = 1mm^3 = 10^{-9}m^3$$

<u>List of constants:</u>
Density of water 1000 kg/m^3;
Viscosity of water $1 \cdot 10^{-3}$ Pa*s
Density of glycerin 1260 kg/m^3,
Viscosity of glycerin 1.5 Pa*s